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THE SPEED OF CONVERGENCE OF A MARTINGALE

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ABSTRACT

Let $X_n, n \geqq 0$, be a martingale with respect to the σ-fields $\mathcal{F}_n$ and let $B_n^2 = \Sigma_{l \geqq n} E\{(X_l - X_{l-1})^2 \mid \mathcal{F}_{l-1}\}$. It is known that if $B_1^2 < \infty$ on some set Ω_0 then $X_\infty = \lim X_n$ exists and is finite a.e. on Ω_0. We show that under suitable conditions there exists a constant $\eta < \infty$ for which $\limsup B_n^{-1}\{\log\log B_n^{-2}\}^{-\frac{1}{2}} |X_\infty - X_{n-1}| \leqq \sqrt{2(\eta+1)}$. If "the fluctuations of B_n are small" (in the sense of the Corollary) then $\eta = 0$ and the usual upper bound of a law of the iterated logrithm results. This upper bound is not necessarily achieved, though.

1. Introduction and statement of results

Throughout $X_n, n \geqq 0$, is a fixed square integrable martingale with respect to the increasing family of σ-fields $\{\mathcal{F}_n\}_{n \geqq 0}$. We denote the corresponding martingale difference sequence by

$$Y_n = X_n - X_{n-1}, \qquad n \geqq 1,$$

and set

$$v_n^2 = E\{Y_n^2 \mid \mathcal{F}_{n-1}\}, \qquad B_n^2 = \sum_{l=n}^{\infty} v_l^2.$$

Stout's law of the iterated logarithm ([11], see also [12], corol. 5.4.2 and p. 303) states that

$$\limsup_{n\to\infty} \left\{ \left(\sum_1^n v_l^2 \right) \log\log \left(\sum_1^n v_l^2 \right) \right\}^{-\frac{1}{2}} X_n = \sqrt{2}, \tag{1.1}$$

if $B_1^2 = \infty$ and some additional hypotheses hold. Since martingales can be imbedded in Brownian motion this result is closely related to the law of the iterated logarithm for a Wiener process $\{W(t)\}_{t \geqq 0}$:

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$$\limsup_{t\to\infty} (t \log\log t)^{-\frac{1}{2}} W(t) = \sqrt{2} \quad \text{w.p.1.} \tag{1.2}$$

(Compare the proof the law of the iterated logarithm for i.i.d. random variables in [2], theorem 13.25 and corollary 12.33.) It is well known that Brownian motion also satisfies a law of the iterated logarithm for small times, i.e. for each fixed s

$$\limsup_{t\to s} \{|t-s| \log\log |t-s|^{-1}\}^{-1/2}(W(t)-W(s)) = \sqrt{2} \quad \text{w.p.1} \tag{1.3}$$

(see [2], theorem (12.29)). The theorem below can be viewed as a partial analogue of (1.3) for martingales for which $B_1^2 < \infty$ on a set of positive probability.

If the increments Y_n are actually independent, better results have already been proven by Chow and Teicher [3] and Barbour [1]. The martingale case seems to have been considered first by Heyde [6]. Heyde's conditions imply all our conditions except for (1.5), after a suitable truncation of the Y_n's (see "Special Case" below). Example (i) below describes a situation where Heyde's theorem does not apply, but where one still has an iterated logarithm bound. Nevertheless, one will normally try to apply Heyde's theorem first; his conditions are less cumbersome and his conclusions considerably stronger than ours.

THEOREM. *Let X_n, Y_n, $\mathcal{F}_n$, v_n^2 and B_n^2 be as above. Assume that there exist constants $0<\eta_1<1$, $\eta_2>0$, a sequence of $\{\mathcal{F}_n\}$ stopping times T_n which increase to ∞ w.p.1, a sequence of random variables a_n and a set $\Omega_0 \in \bigvee \mathcal{F}_n$ such that the following conditions (1.4)–(1.8) hold a.e. on Ω_0:*

$$B_1^2(\omega) < \infty, \tag{1.4}$$

$$Y_n(\omega) = o(B_n(\omega)\{\log\log B_n^{-2}(\omega)\}^{-\frac{1}{2}}), \tag{1.5}$$

$$a_n \text{ is } \mathcal{F}_{T_n} \text{ measurable, } a_n \geqq 0 \text{ and } \liminf_{n\to\infty} \frac{1}{n}\log\frac{1}{a_n} > 0, \tag{1.6}$$

$$a_n^2(\omega) \leqq B_{T_n}^2(\omega) \leqq a_n^2(\omega)\exp(n^{\eta_1}) \text{ eventually} \tag{1.7}$$

and

$$B_{T_{n+1}}^2(\omega) \geqq n^{-\eta_2} B_{T_n}^2(\omega) \text{ eventually.} \tag{1.8}$$

Then $X_\infty = \lim X_n$ exists a.e. on Ω_0 and[†]

[†] We interpret the left hand side of (1.9) as zero when both numerator and denominator vanish for large n. Since $Y_l = 0$ for all $l \geqq n$ a.e. on the set $\{B_n = 0\}$, (1.9) is trivial on the set $\Omega_1 = \{\omega : B_n(\omega) = 0$ for some $n\}$. Thus we can remove Ω_1 from Ω_0 and in the proof in section 2 we tacitly assume $\Omega_0 \cap \Omega_1 = \varnothing$.

$$\limsup_{n\to\infty} \frac{|X_\infty - X_{n-1}|}{B_n\{\log\log B_n^{-2}\}^{\frac{1}{2}}} = \limsup_{n\to\infty} \frac{\left|\sum_n^\infty Y_l\right|}{B_n\{\log\log B_n^{-2}\}^{\frac{1}{2}}} \tag{1.9}$$

$$\leqq \sqrt{2(\eta_1+\eta_2+1)} \text{ a.e. on } \Omega_0.$$

COROLLARY. *If* (1.4)–(1.8) *hold a.e. on* Ω_0 *for all* $\eta_1, \eta_2 > 0$, *then*

$$\limsup_{n\to\infty} \frac{|X_\infty - X_{n-1}|}{B_n\{\log\log B_n^{-2}\}^{\frac{1}{2}}} \leqq \sqrt{2} \tag{1.10}$$

a.e. on Ω_0.

SPECIAL CASE. Assume that for some sequence of stopping times $T_n \uparrow \infty$ and constants $b_n > 0$, $K > 0$ we have

$$P\left\{b_n^2 \leqq B_{T_n}^2 - B_{T_{n+1}}^2 = \sum_{T_n \leqq l < T_{n+1}} v_l^2 \leqq K b_n^2\right\} = 1, \tag{1.11}$$

and as $n \to \infty$

$$\max_{T_n < l \leqq T_{n+1}} |Y_l| = o\left\{\sum_{n+1}^\infty b_l^2\right\}^{\frac{1}{2}}\left\{\log\log\left(\sum_{n+1}^\infty b_l^2\right)^{-1}\right\}^{-\frac{1}{2}} \quad \text{w.p.1.} \tag{1.12}$$

If in addition for all $\eta > 0$

$$\sum_1^\infty b_n^2 < \infty, \quad \liminf \frac{1}{n} \log\left\{\sum_n^\infty b_l^2\right\}^{-1} > 0, \tag{1.13}$$

$$\text{and } \sum_{n+1}^\infty b_l^2 \geqq n^{-\eta} \sum_n^\infty b_l^2 \text{ eventually,}$$

then (1.10) holds w.p.1. (Merely take $a_n^2 = \Sigma_n^\infty b_l^2$.)

This case often applies with $K = 1$ and some b_n^2 if v_n^2 is non-random. In the case where the Y_n are independent Chow and Teicher [3, theorem 2] have proven (1.10) with equality, even without the condition (1.12). Heyde [6] also obtains equality in (1.10), but we already pointed out that his conditions imply ours after a truncation with the exception of (1.5). Indeed, in [6] theorem 1(b)

$$s_n^2 = E\left\{\sum_{l=n}^\infty Y_l^2\right\} < \infty,$$

and b(iii) in [6] implies that for each $\varepsilon > 0$

$$|Y_l| < \varepsilon s_l \text{ eventually and} \tag{1.14}$$

$$\frac{1}{s_n}\sum_{l=n}^\infty E\{Y_l I[|Y_l| < \varepsilon s_l \mid \mathcal{F}_{l-1}\} \to 0 \quad \text{w.p.1}$$

(note that we call X_n what [6] calls Y_n and vice versa). Under the conditions of [6] we can now take

$$a_n = 2^{-n}, \qquad T_n = \sup\{k : s_k^2 \geqq 2^{-n}\}.$$

Then for

$$\tilde{Y}_l(\varepsilon) = Y_l I[|Y_l| < \varepsilon s_l] - E\{Y_l I[|Y_l| < \varepsilon s_l] \mid \mathscr{F}_{l-1}\}$$

one has

$$(1.15)\qquad \begin{aligned} &P\left\{\sum_{T_n \leqq l < T_{n+1}} [\tilde{Y}_l^2(\varepsilon) - E\{\tilde{Y}_l^2(\varepsilon) \mid \mathscr{F}_{l-1}\}] \geqq \varepsilon' 2^{-n}\right\} \\ &\leqq (\varepsilon')^{-2} 2^{2n} \sum_{T_n \leqq l < T_{n+1}} E\{\tilde{Y}_l^4(\varepsilon)\}. \end{aligned}$$

By b(iv) of [6] the sum of (1.15) over n converges. This together with b(i) of [6] in the a.s. version and (1.14) imply

$$2^{n+1} \sum_{T_n \leqq l < T_{n+1}} E\{\tilde{Y}_l^2(\varepsilon) \mid \mathscr{F}_{l-1}\} \to \eta \quad \text{w.p.1}$$

(η as in [6]). Thus, for any sequence $\delta_n \downarrow 0$

$$\delta_n 2^{-n-1} \leqq \sum_{T_n \leqq l < T_{n+1}} E\{\tilde{Y}_l^2(\varepsilon) \mid \mathscr{F}_{l-1}\} \leqq \delta_n^{-1} 2^{-n-1}$$

eventually.

Thus under the conditions of theorem 1(b) in [6] a simple variant of (1.11) and (1.13) apply to the $\tilde{Y}_l(\varepsilon)$.

EXAMPLES. (i) To illustrate the relation with [6] further we consider the Polya urn example of [6], sect. 3. At stage n there are b_n (r_n) black (red) balls in an urn. A ball is drawn at random and replaced and then a random number c_n of balls of the color drawn are added to the urn. If $\mathscr{F}_n$ is an increasing sequence of σ-fields such that b_n, r_n and c_n are $\mathscr{F}_n$ measurable, then $X_n = b_n/(b_n + r_n)$ is a martingale. Heyde considers only the case where $c_n = c$, a constant. It is not hard to vary c_n in such a way that b(i) of [6] fails. In this example

$$|Y_n| \leqq K_n, \qquad \text{and}$$

$$v_n^2 = X_{n-1}(1 - X_{n-1})K_n^2 \sim X_\infty(1 - X_\infty)K_n^2,$$

where

$$K_n = \frac{c_{n-1}}{b_{n-1} + r_{n-1} + c_{n-1}}$$

(see [6], formula (22)). If $1 \leqq c_n \leqq C$ w.p.1, then

$$\frac{1}{b_0 + c_0 + nC} \leqq K_n \leqq \frac{C}{b_0 + c_0 + n} .$$

On the set

$$0 < X_\infty < 1 \tag{1.16}$$

one has eventually for any $\varepsilon > 0$

$$\tfrac{1}{2} X_\infty(1 - X_\infty) C^{-2} n^{-1} \leqq B_n^2 \leqq (1 + \varepsilon) X_\infty (1 - X_\infty) C^2 n^{-1}.$$

Taking $T_n = 2^n$, $a_n^2 = n^{-1} 2^{-n}$ we now easily obtain

$$\limsup_{n \to \infty} (n / \log \log n)^{1/2} |X_n - X_\infty| \leqq C(2X_\infty(1 - X_\infty))^{1/2}$$

a.e. on the set (1.16). Note that

$$E\{(X_\infty - X_{n-1})^2 \mid \mathcal{F}_{n-1}\} = E\{B_n^2 \mid \mathcal{F}_{n-1}\} \leqq 2C^2 n^{-1}$$

from which one easily deduces that (1.16) always has a positive probability when $b_0 > 0$, $r_0 > 0$. With some more work one actually can shown that (1.16) holds w.p.1.

(ii) Another example is obtained by taking $X_n = W(\tau_n)$, where $W(\cdot)$ is a Wiener process as above and τ_n an increasing sequence of stopping times for $W(\cdot)$. Assume that

$$c_n^2 \leqq \tau_n - \tau_{n-1} \leqq K' c_n^2 \quad \text{w.p.1} \tag{1.17}$$

for some constants K', $c_n > 0$ satisfying

$$\begin{gathered} \sum_1^\infty c_n^2 < \infty, \\ c_n^2 \log c_n^{-1} = o \left\{ \sum_{l=n}^\infty c_l^2 \Big/ \left(\log \log \left(\sum_{l=n}^\infty c_l^2 \right)^{-1} \right) \right\} . \end{gathered} \tag{1.18}$$

Then

$$\limsup_{n \to \infty} \frac{|W(\tau) - W(\tau_n)|}{\{(\tau - \tau_n) \log \log (\tau - \tau_n)^{-1}\}^{1/2}} \leqq \sqrt{2K'} \quad \text{w.p.1}.$$

Indeed $c_n^2 \leqq v_n^2 \leqq K' c_n^2$ by (1.17) and since

$$\sum_{l=n+1}^\infty c_l^2 \sim \sum_{l=n}^\infty c_l^2$$

we can take

$$T_n = \sup\left\{k : \sum_k^\infty c_l^2 \geqq (2K')^{-2n}\right\}.$$

Then

$$(2K')^{-2n} \leqq B_{T_n}^2 \leqq (2K')^{-2n+1}$$

eventually, so that (1.11) holds with $b_n^2 = (2K')^{-2n}(1 - 1/2K')$, $K = 2K'/(2K'-1)$. Also (1.12) holds, since by the uniform Hölder condition for the Wiener process

$$\begin{aligned} |Y_l| = |W(\tau_l) - W(\tau_{l-1})| &= O\{(\tau_l - \tau_{l-1})\log(\tau_l - \tau_{l-1})^{-1}\}^{1/2} \\ &= O\{c_l(\log c_l^{-1})^{1/2}\}. \end{aligned}$$

This result gains in interest when it is compared with the results of Orey and Taylor in [10]: there is w.p.1, an everywhere dense set $S \subset [0,\infty)$ of Hausdorff dimension one, such that for all $s \in S$,

$$\limsup_{t \uparrow s} \frac{|W(s) - W(t)|}{\{|s-t|\log\log|s-t|^{-1}\}^{\frac{1}{2}}} = \infty.$$

Apparently if (1.17), (1.18) hold, either $\tau \notin S$ or if $\tau \in S$, τ_n misses the points where $|W(\tau) - W(\tau_n)|$ is much bigger than $\{(\tau - \tau_n)\log\log(\tau - \tau_n)^{-1}\}^{\frac{1}{2}}$.

COUNTEREXAMPLES. The following examples show some of the inherent limitations on weakening the hypotheses or proving a converse of our theorem.

(i) *This example shows that* (1.9) *is essentially sharp.* Let $0 < \beta < 2\gamma$, $\sigma_k^2 \downarrow 0$ and L_k be constants such that L_k is an integer and

$$\sigma_k^2 L_k^2 = (k!)^{-\gamma}, \qquad k^{-\gamma-1}L_k \to 1. \tag{1.19}$$

Now take $X_0 = T_0 = 0$ and define Y_l and T_k such that $T_k - 1$ is a stopping time as follows: On the set $\{T_k - 1 \leqq l < T_{k+1} - 1\} \in \mathscr{F}_l$, the conditional distribution of Y_{l+1}, given $\mathscr{F}_l$ will be normal with mean zero and variance σ_k^2. Moreover,

$$\begin{aligned} T_{k+1} - 1 = \inf\Big\{n \geqq T_k + L_k k^{\frac{1}{2}\beta}(\log k)^{\frac{1}{2}(\beta+3)} &: \sum_{n-L_k+1}^{n} Y_l \\ &\geqq \{(\beta+2)L_k\sigma_k^2 \log\log(L_k\sigma_k^2)^{-1}\}^{\frac{1}{2}}\Big\}, \end{aligned} \tag{1.20}$$

provided there exists an $n \leqq T_k + 2L_k k^{\frac{1}{2}\beta}(\log k)^{\frac{1}{2}(\beta+3)}$ with this property. Otherwise take

$$T_{k+1} = T_k + [2L_k k^{\frac{1}{2}\beta}(\log k)^{\frac{1}{2}(\beta+3)}] + 1. \tag{1.21}$$

In other words, after $T_k - 1$ we add a large number of independent normal random variables with mean zero and variance σ_k^2. We try to add just so many of these random variables that the sum of the last L_k of them is exceptionally large.

By definition $v_l^2 = \sigma_k^2$ for $T_k \leqq l < T_{k+1}$ so that

$$B_{T_k}^2 = \sum_{j=0}^{\infty} (T_{k+j+1} - T_{k+j})\sigma_{k+j}^2. \tag{1.22}$$

Also, by definition

$$L_k k^{\frac{1}{2}\beta}(\log k)^{\frac{1}{2}(\beta+3)} \leqq T_{k+1} - T_k \leqq 2L_k k^{\frac{1}{2}\beta}(\log k)^{\frac{1}{2}(\beta+3)} + 1. \tag{1.23}$$

(1.22), (1.23) and (1.19) show that

$$k^{\frac{1}{2}\beta-\gamma-1}(\log k)^{\frac{1}{2}(\beta+3)}(k!)^{-\gamma} \sim L_k k^{\frac{1}{2}\beta}(\log k)^{\frac{1}{2}(\beta+3)}\sigma_k^2 \leqq B_{T_k}^2$$
$$\leqq 3k^{\frac{1}{2}\beta-\gamma-1}(\log k)^{\frac{1}{2}(\beta+3)}(k!)^{-\gamma}$$

from which (1.4), (1.6)–(1.8) with any $\eta_1 > 0$, $\eta_2 > \gamma$ and

$$a_n^2 = \tfrac{1}{2} n^{\frac{1}{2}\beta-\gamma-1}(\log n)^{\frac{1}{2}(\beta+3)}(n!)^{-\gamma}$$

immediately follow. (1.5) is also easy and thus, by (1.9)

$$\limsup_{n\to\infty} \frac{|X_\infty - X_{n-1}|}{B_n\{\log\log B_n^{-2}\}^{\frac{1}{2}}} \leqq \sqrt{2\gamma+2}. \tag{1.24}$$

On the other hand, simple probability estimates show

$$P\{X_{T_{n+1}-1} - X_{T_{n+1}-L_n-1} \geqq \{(\beta+2)L_n\sigma_n^2 \log\log(L_n\sigma_n^2)^{-1}\}^{\frac{1}{2}} \mid \mathcal{F}_{T_n-1}\} \geqq Kn^{-1} \tag{1.25}$$

for some constant $K = K(\beta,\gamma) > 0$. From (1.25), (1.24), (1.19) and the generalized Borel–Cantelli lemma ([9], corol. VII.2.6) it can be shown that

$$\limsup_{n\to\infty} \frac{|X_\infty - X_{n-1}|}{B_n\{\log\log B_n^{-2}\}^{\frac{1}{2}}} \geqq \limsup_{n\to\infty} \frac{|X_{T_{n+1}-1} - X_{T_{n+1}-L_n-1}|}{\{L_n\sigma_n^2 \log\log(L_n\sigma_n^2)^{-1}\}^{\frac{1}{2}}}$$
$$\geqq \sqrt{\beta+2} \quad \text{w.p.1.} \tag{1.26}$$

Since β can be chosen as close to 2γ as desired, (1.24) and (1.26) substantiate our claim.

(ii) *Even if* (1.4)–(1.8) *hold for any* $\eta_1, \eta_2 > 0$ *for almost all* ω *we may have*

$$\limsup_{n\to\infty} \frac{|X_\infty - X_{n-1}|}{B_n} \leqq C_0 \quad \text{w.p.1,} \tag{1.27}$$

where C_0 is a certain finite constant. We shall not give the lengthy construction of such an example here. It relies on Kahane's solution [8] of Dvoretzky's conjecture ([4]) about times where the law of the iterated logarithm fails for a Brownian motion $W(\cdot)$. Kahane shows that there exists a constant $C_1 < \infty$ and w.p.1 an everywhere dense set S such that for all $s \in S$

$$\limsup_{t \uparrow s} |s - t|^{-\frac{1}{2}} |W(t) - W(s)| \leqq C_1. \tag{1.28}$$

(Kahane even proves (1.28) for the lim sup as $t \to s$, irrespective of $t < s$ or $t > s$.) Kahane does not construct any s satisfying (1.28) as an (accessible) stopping time. To obtain an example satisfying (1.4)–(1.8) for all $\eta_1, \eta_2 > 0$, as well as (1.27), we constructed a sequence of stopping times τ_n for $W(\cdot)$ such that τ_n strictly increases to a limit s which satisfies (1.28), and such that for all $\eta > 0$ one has eventually

$$s - \tau_{n+1} \geqq n^{-\eta}(s - \tau_n).$$

Thus, in general, it is not necessary that

$$\limsup_{n \to \infty} \frac{|X_\infty - X_{n-1}|}{B_n f(n)} > 0 \tag{1.29}$$

for any sequence of normalizing constants $f(n) \uparrow \infty$. We have no simple modification of (1.4)–(1.8) which guarantees (1.29). Note, however, that [6] has a conclusion which is stronger than (1.29).

2. Proof of theorem

It is known (see [9], proposition VII. 2.3 c) that (1.4) implies the convergence to a finite limit of X_n a.e. on Ω_0. Thus we only have to prove (1.9). For typographical convenience we shall often write $B(S)$ instead of B_S. Fix $\theta > 1$ and $0 < \varepsilon < 1$ and introduce the following stopping times[†]:

$$S(n, r, 0) = T_n,$$

$$S(n, r, k+1) = T_{n+1} \wedge \inf\left\{ t > S(n, r, k) : \sum_{l = S(n,r,k)}^{t} v_l^2 \geqq \varepsilon a_n^2 \theta^r n^{-\eta_2} \right\}, \tag{2.1}$$

$$r = 0, 1, \cdots, [(\log \theta)^{-1} n^{\eta_1}], \quad k = 0, 1, \cdots, 2[\theta \varepsilon^{-1} n^{\eta_2}] + 2.$$

Set $\eta = \eta_1 + \eta_2 + 1$ and assume that

$$\sum_{m}^{\infty} Y_l(\omega) \geqq \{\sqrt{2\eta}(1 - \varepsilon)^{-1} + 4\sqrt{\varepsilon\eta\theta}\} B_m(\omega) \left\{ \log\log \frac{1}{B_m^2(\omega)} \right\}^{\frac{1}{2}} \tag{2.2}$$

[†] $a \wedge b$ denotes the minimum of a and b.

for some $\omega \in \Omega_0$ and

$$T_n \leqq m < T_{n+1}. \tag{2.3}$$

Assume further that for this ω and n (1.4) as well as the inequalities in (1.7) and (1.8) hold. Then

$$B^2_{T_{n+1}}(\omega) \geqq n^{-\eta_2} B^2_{T_n}(\omega), \tag{2.4}$$

and there exists an $0 \leqq r \leqq \{\log \theta\}^{-1} n^{\eta_1}$ for which

$$a_n^2 \theta^r \leqq B^2(T_n) < a_n^2 \theta^{r+1}. \tag{2.5}$$

Moreover, if $S(n, r, j+2) < T_{n+1}$, then

$$\sum_{S(n,r,j) \leqq l < S(n,r,j+2)} v_l^2 \geqq \sum_{S(n,r,j) \leqq l \leqq S(n,r,j+1)} v_l^2$$

$$\geqq \varepsilon a_n^2 \theta^r n^{-\eta_2},$$

so that either

$$S(n, r, 2j) \geqq T_{n+1} \tag{2.6}$$

or

$$a_n^2 \theta^{r+1} > B^2(T_n) \geqq \sum_{T_n \leqq l \leqq S(n,r,2j)} v_l^2 \geqq j \varepsilon a_n^2 \theta^r n^{-\eta_2}. \tag{2.7}$$

Clearly (2.7) is impossible for $j > \theta \varepsilon^{-1} n^{\eta_2}$ and (2.6) must hold for any such j. Thus, by (2.3) there must exist a $k \leqq 2\theta\varepsilon^{-1} n^{\eta_2} + 2$ with

$$S(n, r, k) \leqq m < S(n, r, k+1). \tag{2.8}$$

Of course (2.3) and (2.8) together imply $T_n \leqq S(n, r, k) < T_{n+1}$ and hence, by (2.4) and (2.5)

$$a_n^2 \theta^{r+1} > B^2(T_n) \geqq B^2(S(n, r, k)) \geqq B^2(T_{n+1}) \geqq B^2(T_n) n^{-\eta_2} \geqq a_n^2 \theta^r n^{-\eta_2}. \tag{2.9}$$

Consequently, there must exist an integer

$$s \in [0, (\log \theta)^{-1} \eta_2 \log n] + 2$$

for which

$$a_n^2 \theta^{r+s} n^{-\eta_2} \leqq B^2(S(n, r, k)) < a_n^2 \theta^{r+s+1} n^{-\eta_2}. \tag{2.10}$$

Now, by virtue of (1.5), there exists a (deterministic) sequence $\delta_\nu \downarrow 0$ such that a.e. on Ω_0

$$|Y_l(\omega)| \leqq \delta_\nu B_l(\omega)\left\{\log\log\frac{1}{B_l^2(\omega)}\right\}^{-\frac{1}{2}} \qquad \text{for all } l \geqq T_\nu \tag{2.11}$$

and ν sufficiently large. Let δ_ν be fixed in this way and define for each fixed n, r, s[†]

$$u(x) = x^{\frac{1}{2}}\left\{\log\log\frac{1}{x}\right\}^{-\frac{1}{2}}, \qquad 0 < x < 1,$$

$$Y_l^+ = Y_l^+(\omega, n, r, s) = Y_l(\omega) I[Y_l(\omega) \leqq \delta_n u(a_n^2 \theta^{r+s+1} n^{-\eta_2})],$$

$$Y_l^- = Y_l^-(\omega, n, r, s) = Y_l(\omega) I[Y_l(\omega) \geqq -\delta_n u(a_n^2 \theta^{r+s+1} n^{-\eta_2})].$$

Then for almost all $\omega \in \Omega_0$ for which (2.2), (2.3), (2.8) and (2.10) hold and large n, at least one of the events (2.12)–(2.14) must occur:

$$E(n) = \{\omega \in \Omega_0 : (2.11) \text{ fails for } \nu = n\}, \tag{2.12}$$

(2.13) There exists a $S(n, r, k) < t < S(n, r, k+1)$ with

$$\sum_{S(n,r,k) < l < t} Y_l^- \leqq -2\{\varepsilon\eta a_n^2 \theta^{r+s+1} n^{-\eta_2} \log n\}^{\frac{1}{2}},$$

$$\sum_{S(n,r,k)+1}^{\infty} Y_l^+ = \sum_{S(n,r,k) < l < m} Y_l^- + \sum_{m}^{\infty} Y_l - Y_m I[m = S(n,r,k)]$$

$$\geqq \{\sqrt{2\eta}(1-\varepsilon)^{-1} + 4\sqrt{\varepsilon\eta\theta} - \delta_n\} B_m \left\{\log\log\frac{1}{B_m^2}\right\}^{\frac{1}{2}} \tag{2.14}$$

$$-2\{\varepsilon\eta a_n^2 \theta^{r+s+1} n^{-\eta_2} \log n\}^{\frac{1}{2}}.$$

Note now that for almost all $\omega \in \Omega_0$[††]

$$\lim_{n\to\infty} (\log n)^{-1} \log\log B_{T_n}^{-2} = 1. \tag{2.15}$$

Indeed, by (1.6) and (1.7) for large n

$$\log\log B_{T_n}^{-2} \geqq \log\log a_n^{-2} \exp - n^{\eta_1} \geqq \log n + O(1),$$

and if (1.7) and (1.8) hold for $n \geqq n_0(\omega)$, then

$$B_{T_n}^2 \geqq (n-1)^{-\eta_2} B_{T_{n-1}}^2 \geqq \cdots \geqq \prod_{l=n_0}^{n-1} l^{-\eta_2} B_{n_0}^2(\omega) \geqq \{(n-1)!\}^{-\eta_2}\{(n_0-1)!\}^{\eta_2} a_{n_0}^2$$

[†] $I[\ ]$ denotes the indicator of the event between square brackets.

[††] As explained in an earlier footnote we assume without loss of generality that $\Omega_0 \cap \Omega_1 = \varnothing$.

and

$$\limsup (\log n)^{-1} \log\log B_{T_n}^{-2} \leqq \lim (\log n)^{-1} \log\log \{(n-1)!\}^{\eta_2} = 1.$$

Thus, for large enough n we will have

$$(1-\varepsilon)\log n \leqq \log\log B_{T_n}^{-2} \leqq \log\log B_{T_{n+1}}^{-2} \leqq (1+\varepsilon)\log n$$

and also

$$(2.16) \quad (1-\varepsilon)\log n \leqq \log\log B^{-2}(S(n,r,k)) \leqq \log\log B_m^{-2} \leqq (1+\varepsilon)\log n.$$

For later use we point out that (2.15) and (1.7) imply that also a.e. on Ω_0

$$(2.17) \qquad \lim_{n\to\infty} (\log n)^{-1} \log\log a_n^{-2} = 1.$$

Finally, we introduce

$$A_t^2(\omega) = A_t^2(\omega, n, r, k) = \sum_{S(n,r,k)+1}^{t} v_l^2(\omega),$$

and observe that if

$$(2.18) \qquad S(n,r,k) < t < S(n,r,k+1),$$

then, by virtue of (2.1),

$$(2.19) \qquad A_t^2(\omega, n, r, k) < \varepsilon a_n^2 \theta^r n^{-\eta_2}.$$

It follows that the event (2.13) is contained in the event

$$\Gamma(n,r,k,s) = \Big\{\omega\colon \sum_{S(n,r,k)<l\leqq t} Y_l^-(\omega, n, r, s)$$

$$\leqq -\{\varepsilon\eta a_n^2 \theta^{r+s+1} n^{-\eta_2} \log n\}^{\frac12} - \{\varepsilon a_n^2 \theta^{r+s+1} n^{-\eta_2}\}^{-\frac12} (\eta \log n)^{\frac12} A_t^2(\omega)$$

$$\text{for some } t > S(n,r,k)\Big\}$$

Similarly (2.8)–(2.10) and (2.1) imply

$$B_m^2 \geqq B^2(S(n,r,k)) - \varepsilon a_n^2 \theta^r n^{-\eta_2} \geqq (1-\varepsilon) B^2(S(n,r,k)) \geqq (1-\varepsilon) a_n^2 \theta^{r+s} n^{-\eta_2}$$

and (2.10) implies

$$A_t^2(\omega, n, r, k) \leqq B^2(S(n,r,k)) \leqq a_n^2 \theta^{r+s+1} n^{-\eta_2}.$$

Therefore, if (2.8)–(2.10) and (2.16) hold, the event (2.14) is contained in

$$\Delta(n,r,k,s)=\Big\{\omega:\sum_{S(n,r,k)+1}^{t} Y_l^+(\omega,n,r,s)\geqq\{\sqrt{2\eta}+\sqrt{\varepsilon\eta}\}B(S(n,r,k))(\log n)^{\frac12}$$

$$\geqq\tfrac12(\sqrt{2\eta}+\sqrt{\varepsilon\eta})\{a_n^2\theta^{r+s}n^{-\eta_2}\log n\}^{\frac12}$$

$$+\tfrac12(\sqrt{2\eta}+\sqrt{\varepsilon\eta})\{a_n^2\theta^{r+s+1}n^{-\eta_2}\}^{-\frac12}(\log n)^{\frac12}A_t^2(\omega)$$

$$\text{for some } t>S(n,r,k)\Big\}.$$

It follows from the above arguments that up to a null set

$$\{\omega\in\Omega_0\text{: (2.2) occurs for infinitely many } m\}$$

$$\subset\{E(n)\text{ i.o.}\}\cup\Big\{\bigcup_{r,k,s}\Gamma(n,r,k,s)\text{ occurs for infinitely many } n\Big\} \tag{2.20}$$

$$\cup\Big\{\bigcup_{r,k,s}\Delta(n,r,k,s)\text{ occurs for infinitely many } n\Big\}.$$

To conclude the proof we show that

$$\sum_n\sum_{r,k,s}(P\{\Gamma(n,r,k,s)\}+P\{\Delta(n,r,k,s)\})<\infty \tag{2.21}$$

a.e. on Ω_0, whenever

$$\tfrac12(\sqrt{2\eta}+\sqrt{\varepsilon\eta})^2(1+\varepsilon)^{-1}\theta^{-\frac12}>\eta. \tag{2.22}$$

Since $P\{E(n)$ i.o.$\}=0$ by the choice of δ_ν (see (2.11)), (2.20), (2.21) and the Borel–Cantelli lemma imply

$$\limsup_{m\to\infty} B_m^{-1}\left\{\log\log\frac{1}{B_m^2}\right\}^{-\frac12}\sum_m^\infty Y_l\leqq\sqrt{2\eta}(1-\varepsilon)^{-1}+4\sqrt{\varepsilon\eta\theta}$$

whenever (2.22) holds. Replacing Y_l by $-Y_l$ yields the same inequality for $-\Sigma Y_l$, so that (1.9) follows when we let $\theta\downarrow1$, $\varepsilon\downarrow0$. It remains to prove (2.21). We shall restrict ourselves to estimating $P\{\Delta(n,r,k,s)\}$, the estimate for $P\{\Gamma(n,r,k,s)\}$ being almost the same. For fixed n,r,k,s,

$$E\{Y_l^+\mid\mathcal{F}_{l-1}\}\leqq E\{Y_l\mid\mathcal{F}_{l-1}\}=0\qquad\text{and}$$

$$E\{(Y_l^+)^2\mid\mathcal{F}_{l-1}\}\leqq v_l^2.$$

Thus, $\Sigma_{l=1}^t Y^+_{S(n,r,k)+l}$, $t=0,1,\cdots$, is a supermartingale with respect to the σ-fields $\mathcal{G}_t=\mathcal{F}_{S(n,r,k)+t}$, and if we write Z_t for $Y^+_{S(n,r,k)+t}$, then

$$E\{Z_l^2\mid\mathcal{G}_{l-1}\}\leqq v^2_{S(n,r,k)+l}.$$

Moreover

$$Z_l \leqq c \equiv \delta_n \{a_n^2 \theta^{r+s+1} n^{-\eta_2}\}^{\frac{1}{2}} \{(1-\varepsilon)\log n\}^{-\frac{1}{2}}$$

as soon as

$$\log\log\{a_n^2\theta^{r+s+1}n^{-\eta_2}\}^{-1} \geqq (1-\varepsilon)\log n \tag{2.23}$$

(see the definition of Y_l^+). Now set

$$a = \tfrac{1}{2}(\sqrt{2\eta}+\sqrt{\varepsilon\eta})\{a_n^2\theta^{r+s}n^{-\eta_2}\log n\}^{\frac{1}{2}},$$

$$\lambda = (\sqrt{2\eta}+\sqrt{\varepsilon\eta})(1+\varepsilon)^{-1}\{a_n^2\theta^{r+s+1}n^{-\eta_2}\}^{-\frac{1}{2}}(\log n)^{\frac{1}{2}}$$

and

$$\Phi_c(\lambda) = c^{-2}\{\exp(\lambda c) - 1 - \lambda c\}.$$

Then

$$\begin{aligned} P\{\Delta(n,r,k,s)\} &\leqq P\left\{\sum_{S(n,r,k)+1}^{t} Y_l^+ \geqq a + \frac{\Phi_c(\lambda)}{\lambda} A_t^2 \text{ for some } t > S(n,r,k)\right\} \\ &\leqq P\left\{\bigcup_{t\geqq 1}\left\{\sum_{l=1}^{t} Z_l \geqq a + \frac{\Phi_c(\lambda)}{\lambda}\sum_{l=1}^{t} E\{Z_l^2 \mid \mathcal{G}_{l-1}\}\right\}\right\} \end{aligned} \tag{2.24}$$

as soon as

$$\lambda^{-1}\Phi_c(\lambda) \leqq \tfrac{1}{2}(1+\varepsilon)\lambda. \tag{2.25}$$

(2.23) holds for large n by virtue of (2.17) and the restrictions on r, s, while (2.25) holds for large n because $\lambda c \to 0$. Finally by [9], pp. 154–155 or [12] pp. 299–302, the last member of (2.24) is bounded by

$$\exp - \lambda a = \exp - \tfrac{1}{2}(\sqrt{2\eta}+\sqrt{\varepsilon\eta})^2(1+\varepsilon)^{-1}\theta^{-\frac{1}{2}}\log n.$$

Since the triple (r, k, s) runs through at most

$$\{(\log\theta)^{-1}n^{\eta_1}+1\}\cdot\{2\theta\varepsilon^{-1}n^{\eta_2}+3\}\cdot\{(\log\theta)^{-1}\eta_2\log n+3\}$$

values, we see that for almost all $\omega \in \Omega_0$

$$\sum_{r,k,s} P\{\Delta(n,r,k,s)\} = O(n^{\eta_1+\eta_2}\log n \exp - \tfrac{1}{2}(\sqrt{2\eta}+\sqrt{\varepsilon\eta})^2(1+\varepsilon)^{-1}\theta^{-\frac{1}{2}}\log n).$$

Thus, if (2.22) holds we have indeed

$$\sum_n \sum_{r,k,s} P\{\Delta(n,r,k,s)\} < \infty. \qquad \square$$

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We were led to this investigation by the discussion in Section 1.3 and the proof of proposition 3 in [5]. Our theorem can be used to give a simple alternative proof of proposition 3 in [5] (essentially the same observation is made in [7]). We are grateful to Professor Williams for giving us a copy of [5] before publication.

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